Understanding linear dependence-independence, by tertiary students, through the use of a dynamic computing environment

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ABSTRACT: The latest results of ongoing research on tertiary education students from the University of West Attica (UWA), Piraeus-Athens, Greece are presented in this article. The research concerned understanding systems of linear equations and was based on two pre-existing CDFs. Results demonstrate the development of student understanding of basic concepts of vector spaces and linear dependence into a coherent whole including definitions, processes and, most importantly, interrelations. The work was based on David Tall's three worlds of mathematics theoretical framework, and David Kolb's cycles of experiential learning and qualitative research using a modified actions, processes, objects, schemas (APOS) typology. The work investigated learning obstacles and the means of overcoming them. The multi-representational environment was important, although not without problems. Improved cognitive flexibility, procedural understanding and object knowledge were observed among students.

INTRODUCTION

While linear algebra is a field of mathematics that presents major difficulties for tertiary education undergraduate students in general, linear dependence or independence, as a concept, has the added difficulty of being addressed in a variety of ways [1-3]. Matrix equations, vectors and systems of linear equations present different, but interchangeable ways to approach basic concepts. This offers a chance for multiple representations, while adding a level of difficulty, i.e. each representation is familiar to students and yet is different, leading to misconceptions in pre-existing knowledge. Handling multiple representations of the same concept requires students to overcome their established practices and conceptualisations. The latter work well enough when handling each representation, but are inadequate if an overarching cohesive concept is to be developed. Abstract thinking in advanced subjects is difficult for students and is exacerbated by learning gaps from earlier years [4]. Still, a careful introduction by the teacher can allow the rearrangement of the pre-existing separate concepts into a cohesive, working whole.

The research presented in this article has been part of an ongoing project the aim of which is to improve the teaching of introductory lessons to students at the University of West Attica (UWA), Piraeus-Athens, Greece. The goal is to improve learning outcomes and better support students in learning the subject matter in a manner that allows for deeper understanding and does not hinder their ability to apply this knowledge to specific problems. Digital educational technologies were used as tools in the learning process.

While in previous instances, the results and outcomes were presented of interventions to groups of volunteer students of UWA, in this article are found the results of such an intervention in a regular class of first semester students of civil engineering on an introductory lesson in linear algebra. This was accomplished by introducing changes to the traditional learning environment involving a constructivist design of the learning and a use of digital technologies.

One major question for this research proved to be how to achieve conceptual change through the proper use of cognitive conflict or challenge. In the case of linear dependence or independence, the authors attempt to answer that question in this article.

THEORETICAL FRAMEWORK

Tall's Three Worlds of Mathematics

The basic theoretical concept behind this work is that of Tall's three worlds of mathematics [5]. According to Tall, there are three distinct realms of mathematical thinking, i.e. the embodied, the symbolic and the axiomatic. These three worlds permeate the learning process, but are different in structure and relate to different kinds of thinker/learner.

Since the concepts, processes and problems in the three worlds present a number of difficulties, a model better tuned for learning attainment was used as an intermediary capturing tool. The actions, processes, objects and schemas (APOS) model is compatible with Tall's concepts [6]; it categorises knowledge, and is consistent with cognitive constructivism. It posits that acquiring mathematical knowledge involves mental actions, processes and objects organised into schemas.

Finally, the intervention design was based on Kolb's experiential teaching model [7]. Every intervention was designed to include four stages: concrete experience; reflective observation; conceptualisation/theorising; experimentation and action, leading to a new learning experience. The four stages and their sequence are set, although the new learning experience could vary.

DATA COLLECTION AND RESEARCH METHOD

Methods Based on Discussion, Observation and Assessment

The methods used in this study included in-class interviews/discussions, independent researcher observations and assessment of recorded data. The research interview was a discussion between two people, initiated by the interviewer, with the purpose of acquiring relevant information. The content is specified by the purpose, description and interpretation of the research [8]. There were two types of interview. The initial interview was semi-structured. Interviewees described their earlier educational experiences, their knowledge, their views, their conceptions and interpretations of specific subjects. A number of exploratory questions regarding mathematics and linear algebra covered a substantial range of concepts in order to establish a baseline cognitive level for the student.

A number of micro-interviews also occurred, viz. open, of short duration, taking place sometimes more than once, during sessions. These often accompanied the presentation of a new theory revealing a difficulty in understanding on the part of the student. They lasted for a few minutes; their end depending on clarification of the subject, the end of the session or in rare cases a consensus to revisit the subject on another occasion. The interviews took place in the classroom when it was feasible to do so in the time available. Due to time limitations on the part of the participating students, it was decided that the interviews would take place every time there was a difficulty in understanding the material. Even though there was a opportunity for personal interaction, they mostly took place in a group and participation varied.

With an initial thematic approach, the interviews were driven by the answers of the students, which presaged subsequent questions, thus creating a *vortex* of questioning. Students would often also participate with questions of their own, contributions or examples, asking questions of an interviewee or the group. This allowed students to express their idea of basic concepts and processes in solving problems, and also on the interconnections between concepts.

As Clement opined, the interviews gave the researchers an opportunity to collect data on cognitive processes on the ideas of each subject, based on their own mediation, thus exposing cognitive structures and processes [9]. This kind of communication was often unbalanced between interviewer and the interviewee. Hasty questioning might have led to inaccurate or deceptive information [10]. There was also a chance the interviewers' stance and attitude would bias students away from their actual positions and ideas.

RESULTS

During the initial sessions, which were mainly investigational, the students demonstrated adequate knowledge of topics in linear algebra, i.e. matrices, systems, existence of solutions, dependence/independence of vectors, and so on. A large percentage had little difficulty creating vectors within a co-ordinate system, representing them as ordered pairs of 3-tuples and 4-tuples. They could also produce vectors as linear combinations of the vectors of the normal bases of the spaces R^2 and R^3. They could analyse each vector into its components and determine the vector co-ordinates with respect to a particular base. Moreover, they used suitable co-ordinate systems to plot their examples in R^2 and R^3 spaces. In other words, they appear to have understood the representations of a vector, and could articulate the relationship between ways of representing vectors.

In these sessions, the authors employed activities aimed at the investigation of the level of understanding of concepts, but also of the relations between them, for example see Figure 1. In this way, the capability is offered to determine the degree of conceptual understanding of the concepts, and not just of the algorithmic processes that emerge, which can be carried out in a mechanical way. In individual activities, the students used, as needed, either the definition or their observation about the identification of analogies in the relations of vector co-ordinates, so that they could reach faster conclusions and decide whether vectors are linearly dependent or independent and, if so, which.

Check whether the following sets of vectors are linearly independent or not (do this check in as many ways as possible).

1) $\vec{a}(1,4)$, $\vec{b}(-1,7)$ 2) $\vec{a}(1,2,-3)$, $\vec{b}(0,1,4)$, $\vec{c}(0,0,2)$ 3) $\vec{a}(1,5,2)$, $\vec{b}(2,4,6)$, $\vec{c}(2,10,4)$

Figure 1: A characteristic exercise.

However, the general observation regarding the actions of the students is that, although they apply the definition of the linear dependence and independence, they do not realise that this leads to solving systems of linear equations. They do not report it as a finding during the first few sessions, but they perceive it as an examination process of linear dependence and independence (which, in any case, it is) and they pass it by without drawing the necessary conclusions. They do not extract facts, such as the notion of the equivalence of seemingly independent processes. It appears that their knowledge was compartmentalised and not unified; they did not recognise the parts that comprised unified knowledge.

The fourth session followed the model of guided discovery. Relatively strong guidance was necessary for students to overcome the limitations imposed by their previous restricted knowledge of the topic. The session was not characterised by the emergence of theories and experimentation [3]. Nevertheless, there were a few experiments allowing students to understand the role of numerical changes of the vector co-ordinates (within a vector equation) in the shift of relative positions in three-dimensional space (see Figure 2).

Whenever the 3D vector decomposition application was introduced, an interesting difficulty was observed. In their attempt to approach the system of linear equations as a vector equation, the students had difficulty in using x, y and z as factors, as required by the alternative form of application of the vector equation. Regarding the application, it is essential to note the way it is presented to the user. Thus:

a) the application is based on the principle of constructing a vector x, such that $c_1u + c_2v + c_3w = x$. The whole procedure of analysis ends with a system of linear equations, where the role of the usual unknown quantities x, y, z, is assumed by c_1, c_2, c_3;

b) the application does not check whether the system has solutions or not, but simply states whether there are independent vectors or not. This omission was addressed by prompting the students, so that they would determine if a particular construction is indeterminate or inconsistent.

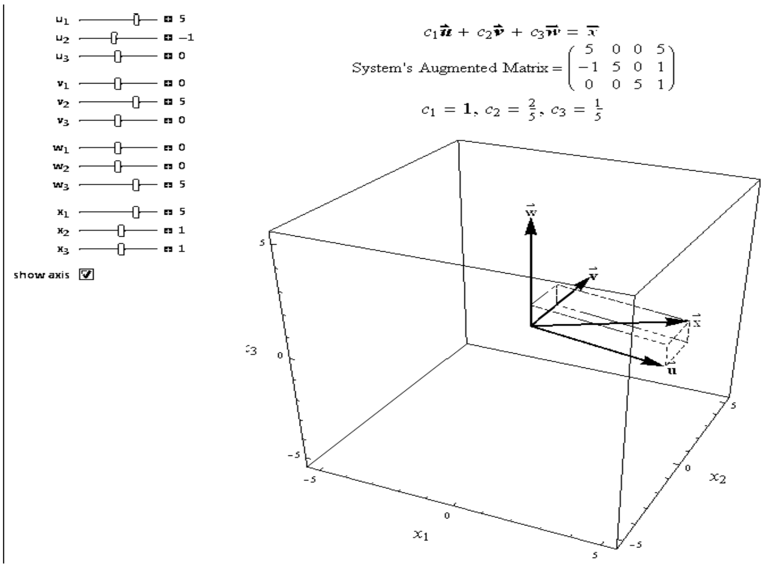

Figure 2: Results of changes of the vector co-ordinates.

The students, although they knew fairly well the respective definitions, initially felt embarrassed. They saw the symbols and the objects, but did not understand their function; even less so that it would assist them in solving the system of equations. They did not have the required familiarisation with the vector expression of a system of linear equations, and could not focus on the essence of the subject. The vector equation created more problems than the existing one. The peculiarity of the symbols resulted in their inability to identify the unknown quantities. It is highly probable that the notion they had about linear dependence and independence did not originate from the understanding of vectors, since this appeared to form during the session itself.

It was through the assistance of the tool that the indirect approach by the students to the topic was revealed; by working with the tool and by experimenting, they eventually approached the concepts and improved their knowledge and perception. The linking of the picture of a system that corresponds to the linear dependence of vectors, to the possibility of construction and checking inside the dynamic environment allowed the transfer of the pre-existing understanding of this concept from the context of R^2, through the appropriate expansion into R^3, and finally to the understanding of the greater variety that this space offers in the cases of linearly dependent vectors.

Initially, the students noticed that the linear dependence in R^3 does not require the vectors to be parallel, but it also adds the novel case of the coplanar vectors, where the linear dependence leads to a unique solution. Consequently, they started to acquire an intuitive grasp of linear dependence and independence in three-dimensional space, transcending the dry application of the strict definition of the independence of a set of vectors. The observation that the equality of vectors requires equality of co-ordinates - in the sense they are not independent equalities, but the simultaneous satisfaction of all equations is required - brought out the concept of a system satisfying many conditions.

At the concept level, the definition of a system is based on the manner in which it is examined. In the present case, the system of linear equations appears prompted by a vector equation. Starting from what was discussed and investigated, the students *saw* the vector x decomposed into the three components of the directions of u, v and w. Subsequently, they interpreted the numerical coefficients c_1, c_2 and c_3 as the multipliers of the vectors that produce the components of x. In other words, based on the abstract relation of the linear combination of vectors, they linked the image of the parallelepiped offered by the dynamic environment and made all the required correlations. The linear dependence of the columns of matrix A (the matrix form of the system) was linked to the uniqueness of the solution. So, the uniqueness of a solution is linked to the number of linearly independent columns, while indeterminate or inconsistent solutions are linked to the number of linearly dependent columns.

The authors observed the development of a more coherent and multi-faceted mental image of the concept of a linear equation system, based on the manner with which the system is examined. At the same time, there was a development of the ability of the students to transfer, through investigative activities, the new knowledge into different contexts. The processes appeared to lead to an embodiment (or at least an encapsulation) of the symbolic knowledge, as predicted by Tall [4]. In this way, the sense is created that an activity which, at first glance, appears unnecessary and even gruelling, ultimately has a significant meta-cognitive value. Through the cognitive conflict that was caused, the picture of the concept of linear system assumes for the students a novel meaning, while around it new conceptual interconnections (linear combination, linear dependence/independence) and new knowledge schemas are developed [11]. The students appeared to fully comprehend, *a posteriori*, the activity.

The benefits for the students stems from the collective negotiation of concepts which, in each particular case, led to further practical exploitation of the environment. There were indications of the benefits of collaboration. The strength of working in groups of two-to-three emerged, when the process took place within a suitably organised learning framework. The process would not operate for certain students without the remarks of their colleagues.

The fifth session was comprised mainly of example-formation activities. The students were asked to produce examples of linear equation systems that would illustrate various relations, such as linear dependence and independence. This was an attempt to combat the difficulties of relating theoretical concepts with geometric content. The introduction of the computing environment led to a change of learning approach from an epistemological point of view, and to the strengthening of the reflection of the students, as they connected the mathematical objects with problem solving [1][12]. The aim of these activities was to trigger reflection in the students, so that they could link definitions and forms, and finally representations to each other.

According to Bogomolny, students who structure examples of matrices with identical rows or with rows that are multiples of one another probably comprehend linear dependence [13]. On the other hand, students who emphasise relations between columns can produce sets of linearly dependent vectors. These latter students are probably assisted by their familiarisation with column relations, which encapsulate linear dependence and, consequently, they are led to a higher degree of abstraction.

At first, students are prompted to offer an example of a 3×3 matrix that has no zero values and with columns linearly dependent. After that, they are prompted to offer an example of a 3×3 matrix that has linearly dependent columns, which have not originated from a vector. The underlying hypothesis for these two activities is that, either because this is the only case they fully understand or because it is the easiest one; the majority of the students would choose for the first example a matrix whose columns can be generated from one column. Of course, in theory the students can present at least one of the following cases:

- a column vector generates the other two columns;
- a column vector is a linear combination of the other two.

The matrix in the first example cannot have a zero column (as the trivial case of linear dependence of a set of vectors), since the students were asked to offer a matrix with no zero values. In this case the authors' hypothesis was confirmed for the vast majority of the participants. Most students indeed responded to the requirements of the first activity by choosing the simplest construction, with vectors being multiples of the first, and hence lying on the same straight line. With this the students appeared to have linear dependence, as connected with a - not necessarily realised - co-linearity. This is further verified by the spontaneity of their choice to express the second and third column as products (multiples) of the first.

The second activity prompted the students to produce an example with a modification involving a different linear dependence between vectors. Here, the third vector was a linear combination of the other two, i.e. the third column was not simply the sum of the other two. Some students, after constructing their own cases of linearly dependent columns, stated the linear dependence through the corresponding vector equation. They wrote down the linear relation between the vectors to reveal the linear dependence. By contrast, other students were satisfied with the matrix and did not write down the linear combination of vectors that demonstrates linear dependence. They eventually did so only after they were prompted. The rest of the students, in spite of all the discussion about linear dependence, did not understand what they should do, and did not produce a suitable matrix, doing so only after explanatory exhortation.

Resorting to the definition in order to prove linear dependence indicates an inability to trust the mechanism by which they produced the vectors, implicitly assuming the geometric argument involving collinear vectors was not enough to demonstrate linear dependence. Vectors were regarded as abstract mathematical objects, without the corresponding geometric visualisation. It is the formal form and easier example that most students related to, due to their previous knowledge and education.

In the following session, students compared the system of equations with the 3D vector decomposition. The three numbers u_1, u_2, and u_3 are the co-ordinates of the first vector, the next three are the co-ordinates of the second vector, and so on. However, the students did not realise that the vectors u, v and w are just the columns of matrix A, if one writes the system in the form Ax = b. The replacement of b with x made the connection even more difficult. The change in the names of the variables certainly created conceptual problems for the students, as is evidenced by the following statement made by a student, viz. *so, v will be the same as y and w the same as z....* This means they equated a vector with a number. The way co-ordinates were introduced caused fellow students to remark that *…they do it in a strange order, it is true*. These students compiled the matrix row by row and not column by column, giving first the first co-ordinates of each vector, then the second ones and lastly the third co-ordinates.

Some students still exhibited a difficulty in understanding the matrix and the vector of constant terms. While some students realised this vector determined the existence of a solution, others were puzzled. Thus, it became necessary to repeat that changes in one coefficient of the system affected the existence and kind of solution, and the meaning of the vectors.

The performance of the subjects and their responses verified a prediction from the literature, i.e. the effectiveness of an intercession using tools and social interactions, as far as the learning process is concerned, usually also depends on the difference in the existing skills of the students [14]. The choice of the spiral character of the educational scheme was also validated.

CONCLUSIONS

The development of the visual-spatial skills of the students was observed in relation to the symbolic, algebraic and matrix approach to the conceptualisation of systems, and the connection between embodied and symbolic knowledge [15]. In Tall's terms, starting from the formal world and having only a formal knowledge of the definitions, students move through to processing concepts in the embodied world, creating cognitive content that allows a symbolic treatment and eventually the return to the formal language with a new depth of knowledge. The indirect approach by the students towards these concepts was pursued and experimented with, eventually allowing the concepts to be approached in a more direct way, thereby improving their knowledge and understanding.

A system that corresponds to a linear dependence of vectors was analysed to allow for the transfer of understanding from R^2, through appropriate generalisation, to R^3. Thus, the concept of linearly dependent vectors was better understood. Students first observed that the linear dependence in R^3 is not restricted to parallel vectors, but also to coplanar vectors. They perceived linear dependence as tantamount to the collapse of a unique solution. Consequently, they acquired an intuitive perception of linear dependence in three-dimensional space, transcending the unproductive application of rigid definitions.

The application caters to representational flexibility, i.e. through particular activities students link the graphical to the algebraic representations and devise solutions to novel problems. However, because there is always the risk of developing their own mistaken theories if left without assistance, suitable counter-examples and problems have been prepared [2][16-18].

The development of visual-spatial skills on the part of the students was noticeable. Starting with a memorised knowledge of the definitions, they processed the concepts, facing cognitive conflicts and creating new interconnections that contributed to the creation of more complex cognitive models and a deeper understanding. Students reached an *objectification* of definitions, which were connected with representations and procedural techniques. Still, the development of a new comprehension of vectors was not the sole benefit of the session. With relative ease, an improved understanding of Gaussian elimination was developed.

The power of processing by a group of students when taking place within a properly organised framework was demonstrated, i.e. processing by some students may not have occurred without the comments of others. The two contributions could be referred to as collaborative creation or, at a higher level, collaborative application. The difference from the use of applications of mathematics was obvious [3].

With the capability to switch between various graphical representations the students could concentrate upon the particular and general properties of objects and created a more integrated and coherent picture of them. The semiotics and syntax of the representational environment was easily achieved. For example, in the sixth session it was possible for a student to input row-by-row despite the initial design of the 3D vector decomposition application for inputting column-by-column. Here was the use of digital technology to improve cognition, which functioned as a reorganiser by presenting mathematical objects in multiple forms and with different, alternative approaches and views [18][19]. Hence, critical thought was strengthened.

Technology offers exquisite mechanisms for visual representations. For example, in the case of calculus, the visualised ideas in the mind need to be connected to the processes of calculation and proof, else they are just intuitive knowledge and not mathematical [5][19]. Multiple applications are, therefore, needed - going through the learning cycle and the *three worlds of mathematics* - in order to achieve a satisfactory level of experience and ability to transfer from one representation to another, a basic characteristic of this teaching.

As a conclusion, it remains as a crucial factor for the acquisition of learning benefit from the multi-representational environment, its pedagogical exploitation and not its simple presence. The theoretical framework set by Tall and the methodology devised by Kolb are tools that allow the maximisation of the benefits of a constructive environment.

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